5.4 Runge-Kutta Methods.

Higher order methods that do not include derivatives of $f(t, y(t))$.

Definition of R-K method requires the use of Taylor's theorem in two Variables.

Theorem 5.3

1) $f(t,y)$ is continuous. on $D = \{(t,y): t \in [a,b], y \in [c,d]\}$
2) All partial derivatives of $f(t,y)$ of order less than or equal to $n+1$ are Continuous on D.

Then, the nth Taylor polynomial in two Variables for the function f about $(t_0, y_0) \in D$ is given by

$$P_n(t,y) = f(t_0,y_0) + (t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)$$

$$+ \frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2} + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) +$$

$$\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0) +$$

$$+\left[\frac{(t-t_0)^3}{3!}\frac{\partial^3 f}{\partial t^3}+3\frac{(t-t_0)^2(y-y_0)}{3!}\frac{\partial^3 f}{\partial t^2\partial y}\right.$$

$$\left.+\;3\;\frac{(t-t_0)(y-y_0)^2}{3!}\frac{\partial^3 f}{\partial t\,\partial y^2}+\frac{(y-y_0)^3}{3!}\frac{\partial^3 f}{\partial y^3}\right](t_0,y_0)$$

$$+\cdots+\left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}\right](t_0,y_0)$$

and

$$f(t,y)=P_n(t,y)+\underbrace{R_n(t,y)}_{\text{Remainder}}$$

Where

$$R_n(t,y)=\left[\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1}f}{\partial t^{n+1-j}\partial y^j}\right](\xi,\mu)$$

Where ξ is btw t and t_0

μ is btw y and y_0.

R-K method of order 2

The starting point in the derivation is Taylor's method of Order 2, for the IVP: $\begin{cases} y'(t) = f(t,y) \\ y(a) = \alpha, \end{cases}$ $a \le t \le b$.

Taylor's thm:

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{3!} y'''(\xi_i)$$

or

$$y_{i+1} = y_i + h f(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i) + \frac{h^3}{3!} y'''(\xi_i)$$

or

$$\boxed{y_{i+1} = y_i + h f(t_i, y_i) + \frac{h^2}{2}\left(f_t(t_i, y_i) + f_y(t_i, y_i) f(t_i, y_i)\right) + \frac{h^3}{3!} y'''(\xi_i).} \quad (3.1)$$

By neglecting the $O(h^3)$ term, we define Taylor's 2nd order:

$$\boxed{w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2}\left(f_t(t_i, w_i) + f_y(t_i, w_i) f(t_i, w_i)\right)} \quad (3.2)$$

with truncation error (3.3)

$$\boxed{\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) - \frac{h}{2}\left(f_t(t_i, y_i) + (f_y \cdot f)(t_i, y_i)\right)}$$

from (3.1)

or $\boxed{\tau_{i+1}(h) = \frac{h^2}{3!} y'''(\xi_i)}$

R-K Midpoint method is defined as

$$\boxed{w_{i+1} = w_i + h f\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right)} \quad (4.1)$$

We want to show that this numerical method is $O(h^2)$.

Proof:- This proof is equivalent to show that replacing the exact solution $y(t)$ into $\frac{(4.1)}{h}$, we get $O(h^2)$ local truncation error:

$$\frac{y_{i+1} - y_i}{h} - f\left(t_i + \frac{h}{2},\ y_i + \frac{h}{2} f(t_i, y_i)\right) = \underbrace{O(h^2)}_{=(\tau_{mp})_{i+1}(h)}. \quad (4.2)$$

To show (4.2) holds, and find an expresion for the local truncation error for the midpoint method $(\tau_{mp})_{i+1}(h)$.

We expand $f(t_i + h/2,\ y_i + \frac{h}{2} f(t_i, y_i))$ in Taylor series about (t_i, y_i).

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Taylor's polynomial in 2-Variables about (t_i, y_i):

$$f\left(t_i+\boxed{h/2},\ y_i+\boxed{h/2\, f(t_i,y_i)}\right)$$

$$= \boxed{f(t_i,y_i) + \frac{h}{2} f_t(t_i,y_i) + \frac{h}{2} f(t_i,y_i) f_y(t_i,y_i)}$$

These three terms match with (3.2)

$$+ \frac{\left(\frac{h}{2}\right)^2}{2} f_{tt}(t_i,y_i) + \frac{2\left(\frac{h}{2}\right)\left(\frac{h}{2} f(t_i,y_i)\right)}{2} f_{ty}(t_i,y_i)$$

$$+ \frac{\left(\frac{h}{2} f(t_i,y_i)\right)^2}{2} f_{yy}(t_i,y_i) + \frac{\left(\frac{h}{2}\right)^3}{3!} f_{ttt}(t_i,y_i)$$

$$+ \cdots$$

$$\tau_{i+1}(h) = \frac{y_{i+1}-y_i}{h} - h\left[f\left(t_i+\frac{h}{2}, y_i+\frac{h}{2} f(t_i,y_i)\right)\right]$$

$$= \frac{y_{i+1}-y_i}{h} - \left[f(t_i,y_i) + \frac{h}{2} f_t(t_i,y_i) + \frac{h}{2} f(t_i,y_i) f_y(t_i,y_i)\right]$$

$$-\left[\frac{h^2}{8} f_{tt}(t_i,y_i) + \frac{h^2}{4} f(t_i,y_i) f_{ty}(t_i,y_i) + \frac{h^2}{8} f^2(t_i,y_i) f_{yy}(t_i,y_i)\right] \quad (5.1)$$

$$+ O(h^3).$$

Now,

$$\frac{y_{i+1}-y_i}{h} - \left[f(t_i,y_i) + \frac{h}{2} f_t(t_i,y_i) + \frac{h}{2} f(t_i,y_i) f_y(t_i,y_i) \right]$$

$$= \frac{h^2}{3!} y'''(t_i) + \cdots = \frac{h^2}{3!} f''(t_i, y_i) + \cdots$$

$$= \frac{h^2}{3!} \Big[f_{tt}(t_i,y_i) + f_{ty}(t_i,y_i) f(t_i,y_i) + f_{yy} f^2(t_i,y_i)$$
$$+ (f_y)^2 f(t_i,y_i) \Big] + \cdots$$

Substitution in the rhs of (5.1)

$$\tau_{i+1}(h) = \frac{h^2}{3!} \left[(f_{tt})_i + (f_{ty} f)_i + (f_{yy} f^2)_i + (f_y^2 f)_i \right]$$
$$- \frac{h^2}{8} \left[(f_{tt}) + 2 (f f_{ty})_i + (f^2 f_{yy})_i \right] + \mathcal{O}(h^3).$$

An alternative way for the derivation of the Midpoint method is (See book page 285 10th Ed.)

Considering the 3 parameters function

$$a_1 f(t+\alpha_1, y+\beta_1) \qquad (5.1)$$

With a_1, α_1, β_1 to be determined. by Comparing with Taylor's second order method.

In order to do this, we expand (5.1) in Taylor polynomial of degree one. In fact,

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t} + \boxed{a_1\beta_1}\frac{\partial f}{\partial y} + a_1 R_1(t+\alpha_1, y+\beta_1). \qquad (5.2)$$

And determine the coeficient comparing with

$$T^2(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t} + \boxed{\frac{h}{2} f}\frac{\partial f}{\partial y}$$

So, $a_1 = 1$, $a_1\alpha_1 = h/2$, $a_1\beta_1 = \frac{h}{2} f(t,y)$

Then, $\boxed{a_1 f(t+\alpha_1, y+\beta_1) = f\left(t+\frac{h}{2}, y+\frac{h}{2} f(t,y)\right)}$

A more general expresion with unknown (4) parameters is given by

$$a_1 f(t,y) + a_2 f(t+\alpha_2, y+\delta_2 f(t,y))$$

by Choosing: $a_1 = a_2 = 1/2$ and $\alpha_2 = \delta_2 = h$

we obtain another 2^{nd} order method:

<u>Modified Euler method:</u>

$$\begin{cases} W_{i+1} = \frac{h}{2}\left[f(t_i, w_i) + f(t_i+h, w_i + h f(t_i, w_i)) \right] \\ W_0 = \alpha \end{cases}$$

or $\boxed{\begin{array}{l} K_1 = h f(t_i, w_i) \\ K_2 = h f(t_i+h, w_i + K_1) \end{array} \Rightarrow W_{i+1} = \frac{1}{2}K_1 + \frac{1}{2}K_2}$

<u>3^{rd} order method:</u> <u>Heun's</u>

$$\begin{cases} W_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3 f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right) \right] \\ W_0 = \alpha \end{cases}$$

To eliminate the successive nesting in the second variable of $f(t,y)$, we define

$$K_1 = hf(t_i, w_i)$$

$$K_2 = hf\left(t_i + \frac{h}{3}, w_i + \frac{1}{3}K_1\right)$$

$$K_3 = hf\left(t_i + \frac{2h}{3}, w_i + \frac{2}{3}K_2\right)$$

Then,

$$\boxed{\begin{aligned} W_{i+1} &= W_i + \frac{1}{4}K_1 + \frac{3}{4}K_3 \\ W_0 &= \alpha \end{aligned}}$$

R-K Order Four

If f has conts derivatives up to order five then, the following method is 4th order.

$$W_0 = \alpha$$

$$K_1 = hf(t_i, W_i)$$

$$K_2 = hf\left(t_i + \frac{h}{2}, W_i + \frac{1}{2}K_1\right)$$

$$K_3 = hf\left(t_i + \frac{h}{2}, W_i + \frac{1}{2}K_2\right)$$

$$K_4 = hf(t_{i+1}, W_i + K_3)$$

$$W_{i+1} = W_i + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

Comment on functional evaluation and how large is the step size for comparison btw methods.